Assignment 1

Philosophical Logic 2025/2026

Instructions

- Discussion among students is allowed, but the assignments should be done and written individually.
- Late submissions will be accepted until one day after the deadline, with a 0.5 penalty.
- Please be explicit and precise, and structure your answers in a way that makes them easy to follow.
- Induction proofs: one or two cases beyond the base step usually suffice.
- Submit a PDF named `PL-2025-A1-⟨your-last-name⟩`.
- For any questions or comments, please contact {m.degano, t.j.klochowicz}@uva.nl
- **Deadline: Tuesday 4 November 2025, 9 pm.**

Conventions. Core notions are as defined in the lecture slides. When asked for a counterexample, provide a specific formula/valuation and verify it by calculation (e.g., a small truth table). Note that when working with Łukasiewicz logics, the constant $1/2$ is *not* given in the language.

Exercise 1 [25 points] Paradox outline

Choose or invent a paradox that fascinates you. Be creative in your presentation: you may choose to illustrate your paradox visually, describe it textually, …

a. List the assumptions/premises and the conclusion that yields the paradox.

b. State the philosophical/logical interest.

c. Either (i) outline a way to dissolve the paradox (what assumption fails; how a reformulation avoids the clash), or (ii) argue briefly why it resists resolution.

(Use no more than 400 words)

Exercise 2 [25 points] Expressivity in Ł$_3$

Recall that in K_3^s and K_3^w we have $p \to q \equiv \neg p \vee q$, but in Ł$_3$ we do not: $p \to q \not\equiv \neg p \vee q$. In this exercise, we explore the expressivity of Ł$_3$. In particular, assume the semantics of $\{\neg, \vee, \wedge, \to\}$ as given in the *Definitions: Many-valued logics - Łukasiewicz* (sec. 1.2.3).

(i) Inexpressibility of $\to$ from $\{\neg, \vee, \wedge\}$. Prove that $\to$ is not definable from $\{\neg, \vee, \wedge\}$ in Ł$_3$. That is, prove that for any formula ϕ whose only sentence letters are p and q and has no other connective besides $\neg$, $\vee$ and $\wedge$, there is a valuation v s.t. $v(\phi) \neq v(p \to q)$.

(ii) Definability of $\vee$ and $\wedge$ from $\{\neg, \to\}$. Show that $\vee$ and $\wedge$ are definable from $\neg$ and $\to$ in Ł$_3$. Find formulas ϕ and ψ whose only sentence letters are p and q and have no other connective besides

$\neg, \to$ such that $\phi \equiv p \vee q$ and $\psi \equiv p \wedge q$. Motivate your answer (e.g., give a truth table, or prove the equivalence). You can assume that $p \vee q \equiv \neg(\neg p \wedge \neg q)$ and $p \wedge q \equiv \neg(\neg p \vee \neg q)$.

Exercise 3 [50 points] Comparing finite Łukasiewicz logics

For $n \geq 2$, let $\mathbf{Ł_n}$ be the n-valued Łukasiewicz logic with

$$T_n = \left\{ \tfrac{k}{n-1} \ : \ k = 0, 1, \ldots, n-1 \right\} \subseteq [0,1]$$

We work with a language over $\neg, \wedge, \vee, \to$ and the usual semantics:

$$\begin{aligned} \neg p &= 1 - p \\ p \wedge q &= \min(p, q) \\ p \vee q &= \max(p, q) \\ p \to q &= \min\big(1,\ 1 - p + q\big) \end{aligned}$$

$\Gamma \vDash_n \phi$ iff for every n-valuation v: if $v(\gamma) = 1$ for all $\gamma \in \Gamma$, then $v(\phi) = 1$.

We aim to establish, for $k, l \geq 2$,

$$\big(\forall \phi \ [\vDash_l \phi \Rightarrow \vDash_k \phi]\big) \iff (k-1) \text{ divides } (l-1)$$

a. [20 points] Sanity checks. We first check a few basic cases.

$$\begin{array}{ll} (1)\ \vDash_3 \phi \ \Rightarrow \ \vDash_4 \phi & (2)\ \vDash_4 \phi \ \Rightarrow \ \vDash_3 \phi \\ (3)\ \vDash_3 \phi \ \Rightarrow \ \vDash_5 \phi & (4)\ \vDash_5 \phi \ \Rightarrow \ \vDash_3 \phi \end{array}$$

(1), (2), and (3) are false, while (4) is true. Provide explicit counterexamples (formula + truth table or calculation) for *each false* statement in (1), (2) and (3). You do *not* need to prove (4) here.

b. [15 points] $\Leftarrow$-direction Prove: if $(k-1)$ divides $(l-1)$ then for all formulas ϕ, $\vDash_l \phi \Rightarrow \vDash_k \phi$.

c. [15 points] $\Rightarrow$-direction-part (i). Prove: if $l < k$, then there exists a formula ϕ such that $\vDash_l \phi$ but $\nvDash_k \phi$.
Hint: use the pigeon-hole principle

d. [Optional, Ungraded, 0 points] Finish the proof and show that if $l \geq k$ and $(k-1)$ does not divide $(l-1)$, there is a formula ϕ with $\vDash_l \phi$ but $\nvDash_k \phi$.